

Optimal randomized incremental construction for guaranteed logarithmic planar point location*

Michael Hemmer[†] Michal Kleinbort[‡] Dan Halperin[‡]

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Abstract

Given a planar map of n segments in which we wish to efficiently locate points, we present the first randomized incremental construction of the well-known trapezoidal-map search-structure that only requires expected $O(n \log n)$ preprocessing time while deterministically guaranteeing worst-case linear storage space and worst-case logarithmic query time. This settles a long standing open problem; the best previously known construction time of such a structure, which is based on a directed acyclic graph, so-called the *history DAG*, and with the above worst-case space and query-time guarantees, was expected $O(n \log^2 n)$. The result is based on a deeper understanding of the structure of the history DAG, its depth in relation to the length of its longest search path, as well as its correspondence to the *trapezoidal search tree*. Our results immediately extend to planar maps induced by finite collections of pairwise interior disjoint well-behaved curves.

The article significantly extends the theoretical aspects of the work presented in <http://arxiv.org/abs/1205.5434>.

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[†]Institute of Operating Systems and Computer Networks, University of Technology Braunschweig, Braunschweig, Germany; mhsaar@gmail.com

[‡]School of Computer Science, Tel Aviv University, Tel Aviv, Israel; balasmic@tau.ac.il, danha@tau.ac.il

1 Introduction

The planar point location problem for a set S of n pairwise interior disjoint x -monotone curves inducing a planar subdivision (or a planar arrangement) $\mathcal{A}(S)$ is defined as follows: given a query point q , locate the feature of $\mathcal{A}(S)$ containing q , i.e., the face, edge or vertex of $\mathcal{A}(S)$ that q lies in. It is one of the fundamental problems in Computational Geometry and has numerous applications in a variety of domains, such as computer graphics, motion planning, computer aided design (CAD), geographic information systems (GIS), and many more.

In this work we revisit one of the most elegant and general algorithms for planar point location, namely the randomized incremental construction of the trapezoidal map and the related search structures. It is also the only algorithm for general x -monotone curves that has an exact, complete and maintained implementation [8, 12], which is available via CGAL, the Computational Geometry Algorithms Library [25].

1.1 Previous Work

As a core problem in Computational Geometry, the planar point location problem has been well-studied for many years. Among the various solutions to the problem, some methods can only provide an *expected* query time of $O(\log n)$ but cannot guarantee logarithmic query time for all cases. It is particularly true for solutions that only require $O(n)$ space. In addition, certain solutions may only support linear subdivisions, while others are applicable to non-linear ones as well. Triangulation-based point location methods, such as Kirkpatrick's approach [13] and Devillers's Delaunay Hierarchy [5] are restricted to linear subdivisions, since they build on a triangulation of the actual input. Kirkpatrick creates a hierarchy of $O(\log n)$ levels of triangulated faces (including the outer face), where at each level an independent-set of low-degree vertices is removed when creating the next level in the hierarchy. This approach guarantees that the data structure requires only $O(n)$ space and that a query takes only $O(\log n)$ time. The *Delaunay Hierarchy* of Devillers, on the other hand, does not guarantee logarithmic query time, and may have a linear query time in the worst case.

Most of the other methods can be summarized under the *trapezoidal search graph* model of computation, as pointed out by Seidel and Adamy [23]. The fundamental search structure used by this model is a directed acyclic graph \mathbb{G} (which may even be just a tree for some methods) with one root and many leaves. Internal nodes in \mathbb{G} have two outgoing edges each, and are either labeled with a vertical line and are therefore left-right nodes, or labeled by an input curve and in such a case are top-bottom nodes. In principal, all these solutions can be generalized to support well-behaved curves [9, Subsection 1.3.3], that is, curves that can be decomposed into a finite number of x -monotone pieces.

One of the earliest solutions that can be subsumed under this model is known as the *slabs method* introduced by Dobkin and Lipton [6]. Every endpoint in-

duces a vertical wall giving rise to $2n + 1$ vertical slabs. A point location query is performed by a binary search to locate the correct slab and another search within the slab in $O(\log n)$ time. Preparata [19] introduced the *Trapezoid Graph method* based on the slabs method. His method, reduces the space bounds from $O(n^2)$, as required by Dobkin and Lipton’s slabs method, to $O(n \log n)$ only, by uniquely decomposing each edge into $O(\log n)$ fragments. Sarnak and Tarjan [20] achieved a significant improvement in memory usage for a slabs-based method by using a *persistent* data structure. Their key observation is that the sequence of search structures in all slabs can be interpreted as one structure that changes over time, which can be stored as a persistent data structure requiring only $O(n)$ size. Another example for this model is the *separating chains method* by Lee and Preparata [15], which also requires linear space. Their algorithm expects a monotone subdivision and uses horizontal monotone chains to separate faces. It is based on the idea that faces of any monotone subdivision can be totally ordered preserving the above-below relation. Each chain is a node in a binary search tree (each edge is kept only once). Querying the structure is essentially deciding whether the query point is above or below $O(\log n)$ chains. However, for each chain this test takes $O(\log n)$, using a binary search. Therefore, the total query time is $O(\log^2 n)$. Another linear size data structure was proposed by Edelsbrunner et al. [7]. They used Fractional Cascading in order to create a layered chain tree as a search structure by copying every other x -value from a node to its parent and maintaining pointers from parent list to child lists. Querying this structure takes $O(\log n)$ time.

This work is focused on the trapezoidal map randomized incremental construction (RIC), which was introduced by Mulmuley [16] and Seidel [21]. Its associated search structure is a Directed Acyclic Graph (DAG) recording the history of the construction. It achieves expected $O(n \log n)$ preprocessing time, expected $O(\log n)$ query time and expected $O(n)$ space. As pointed out by de Berg et al. [4], the latter two can even be guaranteed. However, their sketched solution would require $O(n \log^2 n)$ time. A general major advantage of all variants of this approach is that they can also handle dynamic scenes to some extent, namely, it is possible to add or delete edges later on. The entire method is discussed in more detail in Section 2 below.

In an invited talk [22] at CCCG 2009, Raimund Seidel briefly sketched a deterministic variant with equivalent guarantees that, like the approach of Kirkpatrick, uses independent sets to determine a proper insertion order of segments. However, he also concludes that the elegant and less cumbersome randomized approach, i.e., the RIC, is preferable.

A variant of the latter adds weights and thus gives expected query time satisfying entropy bounds [3]. Arya et al. also stated that entropy preserving cuttings can be used to give a method the query time of which approaches the optimal entropy bound, at the cost of increased space and programming complexity [2]. These methods guarantee a logarithmic query time, however maintaining the search structures requires a considerably large amount of memory and a significant increase in the preprocessing time. Therefore, these solutions are generally rather complicated to implement. For other methods and variants the reader is

referred to a comprehensive overview given in [24].

Contribution

This article extends the theoretical aspects of the work presented in the European Symposium on Algorithms (ESA) 2012 [12], which also presented a major revamp of the exact implementation of the RIC, which now guarantees $O(\log n)$ query time and $O(n)$ space.

Section 2 discusses the basic algorithm by Mulmuley [16] and Seidel [21] and the variant by de Berg et al. [4]. The latter guarantees logarithmic query time by reconstructing the search structure if the length of the longest search path \mathcal{L} or the size \mathcal{S} exceed some thresholds. However, to keep the preprocessing efficient, \mathcal{L} and \mathcal{S} would have to be efficiently accessible, which is not trivial for \mathcal{L} . In fact, an early version of [4] did not make the distinction between \mathcal{L} and the depth \mathcal{D} of the DAG, which is the length of the longest DAG path and can be efficiently accessed. Section 3 discusses the fundamental difference between \mathcal{D} and \mathcal{L} . Specifically, we show that the worst-case ratio between \mathcal{D} and \mathcal{L} can be $\Theta(n/\log n)$. In Section 5 we introduce two algorithms to verify \mathcal{L} after the actual construction, both leading to an overall expected $O(n \log n)$ preprocessing time. The first relies on a deeper understanding of the relation between the trapezoidal search tree \mathbb{T} and the DAG \mathbb{G} (A preparatory discussion of \mathbb{T} and \mathbb{G} is given in Section 4). It operates directly on \mathbb{G} and requires expected $O(n \log n)$ time. The second runs in deterministic $O(n \log n)$ time and is based on the computation of the ply of all trapezoids that existed during the construction of \mathbb{G} . Conclusions and open problems are given in Section 6. We defer to Appendices some auxiliary material including some straightforward case-analysis, description of folklore results that we have not found archived, and adaptation of known algorithms to our specific needs.

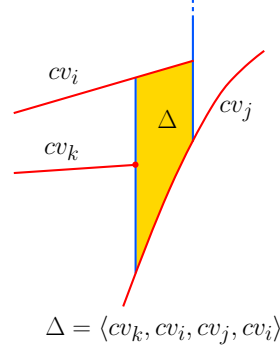
2 Preliminaries

This section briefly reviews the trapezoidal map random incremental construction for point location. After some relevant general definitions in Subsection 2.1, the basic algorithm presented by Mulmuley [16] and Seidel [21] is provided in Subsection 2.2. The variant by de Berg et al. [4], which gives the guarantees on \mathcal{L} and \mathcal{S} , is described in Subsection 2.3.

2.1 Definitions

Let S be a set of n pairwise interior disjoint x -monotone curves in general position, i.e., no two distinct endpoints have the same x -coordinate and no endpoint of one curve lies in the interior of another curve. S induces a planar subdivision (or a planar arrangement) $\mathcal{A}(S)$, which is composed of vertices, and faces, in addition to its n edges.

The *Trapezoidal Map* of an arrangement $\mathcal{A}(S)$, denoted by $\mathcal{T}(S)$, is obtained by extending vertical walls from each endpoint upwards and downwards until an input curve is reached or the wall extends to infinity. Each trapezoid Δ of $\mathcal{T}(S)$ is restricted by at most two curves¹, denoted by $bottom(\Delta)$ and $top(\Delta)$, and also by either one or two vertical walls (trapezoid bases). $left(\Delta)$, $right(\Delta)$ denote the curves whose endpoints induce the left and right vertical walls, respectively. Therefore, each trapezoid Δ in $\mathcal{T}(S)$ can be defined by a unique quadruplet: $\langle left(\Delta), right(\Delta), bottom(\Delta), top(\Delta) \rangle$, as depicted in the figure to the right. The trapezoidal map $\mathcal{T}(S)$ is unique and does not depend on the order of insertion. As shown in [4], $\mathcal{T}(S)$ of an arrangement consisting of n curves has at most $3n + 1$ trapezoids and at most $6n + 4$ vertices.



The trapezoids in $\mathcal{T}(S)$ are neighbors if they share a vertical wall. As we assume general position, each trapezoid has at most four neighboring trapezoids: at most two along its left vertical edge, and the same along its right vertical edge. We remark that the general position assumption poses no restriction on the algorithm as one can use a symbolic shear transformation, i.e., by simply replacing comparisons of x -coordinates by lexicographical xy -comparisons.

For simplicity of presentation, and w.l.o.g., the following figures contain horizontal line-segments.

2.2 The Basic RIC Algorithm

We review the random incremental construction (RIC) of a point location structure, as introduced in [16, 21] and described in [4, 17]. Given an arrangement $\mathcal{A}(S)$ of n pairwise interior disjoint x -monotone curves, a random permutation of the curves is inserted incrementally, constructing the trapezoidal map $\mathcal{T}(S)$. During the incremental construction, an auxiliary search structure, a directed acyclic graph (DAG) \mathbb{G} , is maintained. The DAG \mathbb{G} has one root and many leaves, one for every trapezoid in the trapezoidal map $\mathcal{T}(S)$. Every internal node is a binary decision node, representing either an endpoint p of an input curve, deciding whether a query point q lies to the left or to the right of the vertical line through p , or an x -monotone curve cv_i , deciding whether the query point q is above or below it. When reaching a curve-node representing a curve cv_i , it is guaranteed that the query point q lies in the x -range of cv_i . In addition, the trapezoids in the leaves of \mathbb{G} are interconnected, that is, each trapezoid knows its (at most) four horizontal neighbors (two to the left and two to the right). For a simple example refer to Figure 1(a), where the DAG is still a tree.

¹We use the term *trapezoid* even when the side edges are not linear segments.

2.2.1 Insertion

When a new x -monotone curve is inserted, the trapezoid containing its left end-point is located by a search from root to leaf. Then, using the connectivity information described above, the trapezoids intersected by the curve are gradually revealed and updated. Merging new trapezoids, if needed, takes time that is linear in the number of intersected trapezoids. The merge turns the data structure into a DAG with expected $O(n)$ size [16, 21]. By skipping the merge step, one would obtain a binary tree, known as the *trapezoidal search tree*, having expected $O(n \log n)$ size, as shown in Section 4. The whole insertion process is illustrated in Figure 1. For an unlucky insertion order the size of the resulting data structure may be quadratic, and the longest search path may be linear. However, since the curves are inserted in a random order, one can expect $O(n)$ space, $O(\log n)$ query time, and $O(n \log n)$ preprocessing time. For proofs see [4, 16, 21].

As a result of the merge the search structure may contain more than one valid search path from the root to a certain leaf, as demonstrated in Figure 1(c).

2.3 Previous Attempts at Guaranteeing Logarithmic Query Time

The basic algorithm, described in Subsection 2.2, requires expected $O(n \log n)$ time and expected $O(n)$ space. Moreover, it is not hard to see that the expected query time for an arbitrary but fixed query point is $O(\log n)$. However, de Berg et al. [4] showed that the probability that the length of the longest search path \mathcal{L} is larger than $3\lambda \ln(n+1)$ is rather small, e.g., for $\lambda = 20$ and $n > 4$ it is less than $1/4$. A similar argument can be applied for the size \mathcal{S} of the constructed DAG [14]. This leads to the idea that one can guarantee a linear size data structure with guaranteed logarithmic query time by simply re-constructing the data structure with a new random insertion order until it has the required properties. Essentially de Berg et al. [4] show the following crucial lemma:

Lemma 1. *It is possible to choose suitable constants $c_1, c_2 > 0$ such that the expected number of rebuilds that are required to achieve $\mathcal{S} < c_1 n$ and $\mathcal{L} < c_2 \log n$ is a small constant.*

Not taking the cost for the verification of \mathcal{L} and \mathcal{S} into account this would immediately lead to an algorithm that in total still runs in expected $O(n \log n)$ time; see also Section 5. However, while it is straightforward to keep track of \mathcal{S} , an efficient verification of \mathcal{L} is not trivial at all. Specifically, one should be aware of that \mathcal{L} is not equal to the depth \mathcal{D} of the DAG, i.e., the length of the longest DAG path. Note that \mathcal{L} is determined only by the valid search paths and not by all paths. This subtle difference, which is discussed in Section 3, caused some confusion in the past. Thus, since the sketch of the verification algorithm in the last version of de Berg et al. [4] requires $O(n \log^2 n)$ time, until now no expected $O(n \log n)$ time algorithm giving the above guarantees was known.

3 Depth vs. Maximum Query Path Length

The depth \mathcal{D} of the DAG is an upper bound on \mathcal{L} , as the set of all possible search paths is a subset of all paths in the DAG. \mathcal{D} can be made accessible in constant time, by storing the depth of each leaf in the leaf itself, and by maintaining the maximum depth in a separate variable. The cost of maintaining the depth can be charged to new nodes, since existing nodes never change their depth value. Having said that, it is not clear how to efficiently access \mathcal{L} while retaining linear space, since each leaf would have to store a non-constant number of values, i.e., one for each valid search path that reaches it. In fact, the memory consumption would be equivalent to the trapezoidal search tree, which is expected to be of size $O(n \log n)$, as shown in Appendix A.

We show that the depth \mathcal{D} of a given DAG can be linear while its maximum query path length \mathcal{L} is still logarithmic, that is, such a DAG would trigger an unnecessary rebuild. It is thus questionable whether we can still expect a constant number of rebuilds when relying on \mathcal{D} . Figure 2 demonstrates the difference between the DAG depth \mathcal{D} and the maximum query path length \mathcal{L} .

The following construction, which uses a recursive scheme, establishes the worst-case lower bound $\Omega(n/\log n)$ for \mathcal{D}/\mathcal{L} . There are $\log_2 n$ blocks, where block i contains $n/2^i$ segments. Within each block the same scheme is applied recursively, as depicted in Figure 3. The segments are inserted from top to bottom such that the depth of $\Omega(n)$ is achieved in the trapezoid below the lowest segment. The fact that the lengths of all search paths are logarithmic can be proven by the following argument. By induction we assume that the longest search path within a block of size $n/2^i$ is some constant times $(\log_2 n - i)$. Obviously this is true for a block containing only one segment. Now, in order to reach block i containing $n/2^i$ segments, we require $i - 1$ comparisons to skip the $i - 1$ st preceding blocks. Thus in total the search path is of logarithmic length.

Theorem 2. *The $\Omega(n/\log n)$ worst-case lower bound on \mathcal{D}/\mathcal{L} is tight.*

Proof. Obviously, \mathcal{D} of $O(n)$ is the maximal achievable depth, since by construction each segment can only appear once along *any* path in the DAG. It remains to show that for any scenario with n segments there is no DAG for which \mathcal{L} is smaller than $\Omega(\log n)$. Since there are n segments, there are at least n different trapezoids having these segments as their top boundary. Let T be a decision tree of the optimal search structure in the sense that its longest query path is the shortest possible. Each path in the decision tree corresponds to a valid search path in the DAG and vice versa. The depth of T must be at least $\log_2 n$, since it is only a binary tree. We conclude that the worst-case ratio \mathcal{D}/\mathcal{L} is $\Theta(n/\log n)$. \square

4 A Bijection between the Search Paths in the History DAG and in the Trapezoidal Search Tree

Let S be a set of n pairwise interior disjoint x -monotone curves inducing a planar subdivision. The trapezoidal search tree \mathbb{T} for S is a full binary tree constructed as the DAG \mathbb{G} using the same insertion order while skipping the merge step.

The DAG \mathbb{G} has an expected linear size [16, 21]. On the other hand, the trapezoidal search tree \mathbb{T} requires $\Omega(n \log n)$ memory for certain scenarios, as shown in [23]. The following lemma, which seems to be folklore, bounds the expected size of \mathbb{T} . For completeness, we give a proof in Appendix A.

Lemma 3. *Let S be a set of n pairwise interior disjoint x -monotone curves inducing a planar subdivision. The expected number of leaves in the trapezoidal search tree \mathbb{T} , which is constructed as the DAG but without merges, is $O(n \log n)$.*

As we show next, there is a bijection between all possible search paths in \mathbb{T} and those of \mathbb{G} , even though the two structures differ in size. First, let us define here the notion of *bouncing nodes*. Suppose we query \mathbb{G} with point q . Additionally, assume that while searching for q we maintain an interval of the x -values that are still possible with respect to the decisions taken so far. This *history interval* is updated at each decision node according to the following scheme: (i) if the node is a curve node then the history interval does not change (ii) if the node is a point node whose x -coordinate is contained in the current interval, then the interval is updated according to the position of q (iii) if the node is a point node whose x -coordinate is not contained in the current interval then the interval remains unchanged. Such a point node that is not contained in the current history interval of the path is named a **bouncing-node for the corresponding path** in \mathbb{G} ; Figure 4 gives an example of a bouncing node.

The following proposition shows that each search path in \mathbb{G} has a corresponding path in \mathbb{T} (and vice versa), and these two paths are identical up to additional bouncing nodes in the path in \mathbb{G} .

Proposition 4. *Let S be a set of n pairwise interior disjoint x -monotone curves inducing a planar subdivision. Let \mathbb{G} and \mathbb{T} be the DAG and the trapezoidal search tree created using the same permutation of the curves in S , respectively. There exists a canonical bijection among all search paths in \mathbb{G} and those of \mathbb{T} , that is, for any query point q , the corresponding search paths for q in \mathbb{G} and \mathbb{T} are identical up to bouncing nodes.*

Proof. Let q be a query point and t and t' be the leaf trapezoids containing q in \mathbb{G} and \mathbb{T} , respectively. Obviously, the top and bottom curves of t and t' are identical and t covers t' since merges are only allowed in \mathbb{G} . Suppose that while searching for q we maintain the history-interval of possible x -values. At the end of the search in \mathbb{T} this interval is obviously identical to the x -range of t' . We show by induction the bijection between the two search paths for q and in fact

we also show that the history intervals maintained while searching in \mathbb{G} and \mathbb{T} are identical, i.e., the history interval that is eventually obtained by the search in \mathbb{G} is identical to the x -interval of t' .

Let $\mathbb{G}_i, \mathbb{T}_i$ denote the DAG and the trapezoidal search tree after the first i curves were inserted, respectively. We denote by t_i and t'_i the trapezoids containing the query point q in \mathbb{G}_i and \mathbb{T}_i , respectively. Let (a_i, b_i) and (a'_i, b'_i) denote the x -intervals of t_i in \mathbb{G}_i and t'_i in \mathbb{T}_i , respectively. The base case is trivial since $\mathbb{G}_1 = \mathbb{T}_1$. Now suppose that the statement holds for $i - 1$. We show that it holds for i as well. The i th curve $cv_i(p_i, q_i)$ is now inserted into both \mathbb{G}_{i-1} and \mathbb{T}_{i-1} . The basic argument is as follows. For both endpoints of cv_i there are essentially three cases: (i) the point is outside t_{i-1} and t'_{i-1} : the point has no effect on both paths. (ii) the point is inside t_{i-1} and t'_{i-1} : the point shows up as a normal node in both paths and the (identical) history intervals are updated accordingly. (iii) the point is inside t_{i-1} but not in t'_{i-1} : the point has no effect on the search path in \mathbb{T}_i while it may show up on the search path in \mathbb{G}_i , but only as a bouncing node, i.e., the history interval remains unchanged. Figure 5 shows the 15 possible positions to insert cv_i with respect to t_{i-1} and t'_{i-1} .

As an example we discuss position 13, while the full and rather straightforward case analysis is given in Appendix B. In this case p_i as well as q_i are inside t_{i-1} but to the right of t'_{i-1} . The search path for q in \mathbb{T}_i remains unchanged since t'_{i-1} is not destroyed whereas the path in \mathbb{G}_i changes as t_{i-1} is destroyed. However, the only change is the addition of p_i , which is a bouncing node for that path since it is not contained in the history-interval, i.e., the x -range remains unchanged since $t'_{i-1} = t'_i$. Notice that, in this particular case, the right end point q_i does not even appear as a bouncing node since it is shadowed by p_i . \square

Lemma 5. *Every edge $e' \in \mathbb{T}$ can be associated to precisely one sequence of edges in the corresponding DAG \mathbb{G} .*

Proof. Since \mathbb{T} is a tree, all search paths in \mathbb{T} that use e' are identical up to that point. Thus, the decisions taken at intermediate bouncing nodes while following the corresponding path in \mathbb{G} are predetermined due to their common history. \square

Hence, in the following we say that $e' \in \mathbb{T}$ accumulates bouncing nodes, namely the bouncing nodes on its corresponding subpath in \mathbb{G} .

Observation 6. *Let $\mathbb{G}_{i-1}, \mathbb{T}_{i-1}$ be as defined above. Only edges in \mathbb{T}_{i-1} that currently end in leaves may accumulate additional bouncing nodes due to the insertion of the i th curve. More precisely, let t'_{i-1} be the trapezoid in which the leaf edge e' ends and let t_{i-1} be the trapezoid in \mathbb{G}_{i-1} that covers t'_{i-1} , as illustrated in Figure 5. The edge e' may only accumulate additional bouncing nodes iff t_{i-1} is destroyed.*

Definition 7. *An edge of \mathbb{T} that at some intermediate step of the construction was a leaf edge, that is, ended in a trapezoid, is named a critical edge.*

Note the direct correspondence between a critical edge and its trapezoid. An edge remains a leaf edge until its trapezoid is destroyed, in which case the trapezoid is replaced by an internal node of \mathbb{T} .

Observation 8. *The insertion of a single curve may incur at most two additional bouncing nodes for a search path in \mathbb{G}_i .*

5 Efficient Construction Algorithms for Static Settings

Given a set S of n pairwise interior disjoint x -monotone curves inducing a planar subdivision, we seek to devise an efficient construction algorithm for static settings, when all input curves are given in advance, which results in a linear-size DAG and logarithmic query time in the worst case. Theorem 10 gives resource bounds on such an algorithm based on the availability of an efficient verification algorithm for \mathcal{L} .

Definition 9. *Let $f(n)$ denote the time it takes to verify that, in a linear size DAG constructed over a set of n pairwise interior disjoint x -monotone curves, \mathcal{L} is bounded by $c \log n$ for a constant c .*

Theorem 10. *Let S be a set of n pairwise interior disjoint x -monotone curves inducing a planar subdivision. A point location data structure for S , which has $O(n)$ size and $O(\log n)$ query time in the worst case, can be built in $O(n \log n + f(n))$ expected time, where $f(n)$ is as defined above.*

Proof. The construction of a DAG with some random insertion order takes expected $O(n \log n)$ time. The linear size can be verified trivially on the fly (as discussed in Subsection 2.3). After the construction an algorithm, requiring $f(n)$ time, that verifies that the maximum query path length \mathcal{L} is logarithmic is used. The verification of the size S and the maximum query path length \mathcal{L} may trigger a rebuild with a new random insertion order. However, according to Lemma 1 one can expect only a constant number of rebuilds. Thus, the overall expected running time remains $O(n \log n + f(n))$. \square

The next two subsections describe two efficient verification algorithms for \mathcal{L} that can be used by the general construction algorithm. The first one uses the existing search structure and has expected $O(n \log n)$ running time. The second algorithm is less straightforward to apply but has worst-case $O(n \log n)$ running time.

5.1 An Expected $O(n \log n)$ Verification Algorithm

The following algorithm verifies that the maximum query path length \mathcal{L} in the search structure is bounded by $c_{\mathcal{L}} \log(n)$, where $c_{\mathcal{L}}$ is some properly chosen constant according to [4, Sec 6.4]. The algorithm is recursive, starting at the root it descends towards the leaves and explores all possible search paths and

discards all other paths that are geometrically unrealizable. To do so, each recursion call maintains the history-interval of x -values that are still possible with respect to the decisions taken so far.

The algorithm starts at the root with the maximal interval, i.e., $[-\infty, +\infty]$. At each node there are three possible cases: (i) the recursion reaches a curve node, it splits for the upper and lower path while the interval remains unchanged; (ii) the node is a point node whose x -coordinate is contained in the current interval I , the recursion splits to the left and the right side with updated intervals, i.e., I is split at the x -coordinate of the node; (iii) the node is a point node whose x -coordinate is *not* contained in I (bouncing node for this path), the recursion does not split and continues to the proper child only with I unchanged. Figure 6 illustrates a partial run of the algorithm.

The expected running time of the above recursive algorithm applied to \mathbb{T} would be $O(n \log n)$. This follows from the fact that the algorithm would use each edge of the tree exactly once and by the expected size of the tree, which by Lemma 3 is $O(n \log n)$. In fact, the behavior of the algorithm when applied to the corresponding DAG \mathbb{G} is very similar since the bouncing nodes create additional costs but do not let the recursion split. That is, the algorithm still follows each edge e' of \mathbb{T} but with extra costs per edge incurred by bouncing nodes, see also Lemma 5. Thus, the total cost of the recursive verification algorithm applied to \mathbb{G} is

$$\sum_{e' \in \mathbb{T}} (w_{e'} + 1), \quad (1)$$

where $w_{e'}$ is the number of additional bouncing nodes in the corresponding subpath for e' in \mathbb{G} . By Observation 6 we know that only critical edges may accumulate many bouncing nodes. On the other hand, by Observation 8, all other edges can only accumulate up to two bouncing nodes. Each of the critical edges can be associated with a trapezoid that existed during the construction of \mathbb{T} . Hence, let $\Delta^{\mathbb{T}}$ denote the set of all trapezoids that were created during the construction of \mathbb{T} . For every such trapezoid $t' \in \Delta^{\mathbb{T}}$ we define its weight as $w_{t'} = w_{e'}$, where e' is its corresponding critical edge in \mathbb{T} . For all other edges the total cost is at most 3 (at most 2 bouncing nodes by Observation 8). Hence, we can upper bound (1) as follows:

$$\sum_{e' \in \mathbb{T}} (w_{e'} + 1) \leq 3|\mathbb{T}| + \sum_{t' \in \Delta^{\mathbb{T}}} w_{t'}. \quad (2)$$

Let Δ be the set of all possible trapezoids that may exist during the construction of a trapezoidal search tree. Now, set $w_{t'} = 0$ for all $t' \in \Delta \setminus \Delta^{\mathbb{T}}$, i.e., for those that were not created with respect to a specific insertion order. We can now extend the right hand side of (2) without changing its value as follows:

$$3|\mathbb{T}| + \sum_{t' \in \Delta^{\mathbb{T}}} w_{t'} = 3|\mathbb{T}| + \sum_{t' \in \Delta} w_{t'}. \quad (3)$$

We are interested in the expected value of the right hand side of (3), i.e., the expected value with respect to all $n!$ possible insertion orders of the segments in S . This can be written as

$$O(n \log n) + \mathbb{E}\left[\sum_{t' \in \Delta} w_{t'}\right], \quad (4)$$

since, by Lemma 3, the expected size of \mathbb{T} is $O(n \log n)$. By linearity of expectation (4) is equivalent to:

$$O(n \log n) + \sum_{t' \in \Delta} \mathbb{E}[w_{t'}] \quad (5)$$

Let $\delta_{t'} = 1$ if $t' \in \Delta^{\mathbb{T}}$ and 0 otherwise. By the law of iterated expectation we can now split up the expected value according to the condition whether t' exists during the construction of \mathbb{T} or not.

$$O(n \log n) + \sum_{t' \in \Delta} \left(\mathbb{E}[w_{t'} | \delta_{t'} = 1] \Pr[\delta_{t'} = 1] + \mathbb{E}[w_{t'} | \delta_{t'} = 0] \Pr[\delta_{t'} = 0] \right) \quad (6)$$

Observing $\mathbb{E}[w_{t'} | \delta_{t'} = 0] = 0$ and $\Pr[\delta_{t'} = 1] = \mathbb{E}[\delta_{t'}]$, we are left with:

$$O(n \log n) + \sum_{t' \in \Delta} \mathbb{E}[w_{t'} | \delta_{t'} = 1] \mathbb{E}[\delta_{t'}]. \quad (7)$$

Most of the remainder of the section is dedicated to the fact that $\mathbb{E}[w_{t'} | \delta_{t'} = 1]$ is a constant. It is then straightforward to conclude that the expected running time is $O(n \log n)$; see Proposition 14 at the end of this section.

Let Π be the set of all $n!$ insertion sequences. Every $\pi \in \Pi$ defines a construction of a trapezoidal search tree $\mathbb{T}(\pi)$ and the corresponding DAG $\mathbb{G}(\pi)$. Recall that the difference between the trapezoidal map of \mathbb{T} and \mathbb{G} are the merges that occur during the construction of \mathbb{G} . Hence, a trapezoid $t' \in \Delta^{\mathbb{T}(\pi)}$ may be covered by several trapezoids of \mathbb{G} during its existence. We denote the number of these trapezoids by $n_{t'}(\pi)$. Obviously, $n_{t'}(\pi) = 0$ iff $t' \notin \Delta^{\mathbb{T}(\pi)}$.

Lemma 11. *Let $\Pi^{n_{t'} \diamond i} = \{\pi \in \Pi | n_{t'}(\pi) \diamond i\}$ for $\diamond \in \{=, <, >, \leq, \geq\}$. For any integer $i > 0$, the number of insertion sequences where $n_{t'}(\pi) = i$ is greater or equal to the number of sequences where $n_{t'}(\pi)$ is larger than i , that is:*

$$|\Pi^{n_{t'} > i}| \leq |\Pi^{n_{t'} = i}|.$$

Proof. We first define a map $\phi_{t'}^i : \Pi^{n_{t'} > i} \rightarrow \Pi^{n_{t'} = i}$ and then show that it is well-defined and injective.

Definition of $\phi_{t'}^i$: Since $\pi \in \Pi^{n_{t'} > i}$ there is a sequence of more than i trapezoids from $\Delta^{G(\pi)}$ that cover t' . Let t be the i -th last trapezoid in that sequence. Let $S(t)$ be the set of at most 4 segments that define t . Let s be the segment among those in $S(t)$ that is inserted last with respect to π . Notice that t' must already exist when s is inserted since $n_{t'}(\pi) > i > 0$. Hence, s

cannot be in $S(t')$, otherwise it would contradict the fact that t' already exists. Therefore, s can only be $left(t)$ or $right(t)$, which must extend to the left or right, respectively. If s equals $left(t)$ swap it with $top(t) = top(t')$, otherwise with $bottom(t) = bottom(t')$. Assuming that s was at position j and that the swapped segment \bar{s} was at position $k < j$, the resulting sequence $\phi_{t'}^i(\pi)$ is $[s_1, \dots, s_{k-1}, s, \dots, \bar{s}, s_{j+1}, \dots, s_n]$.

$\phi_{t'}^i$ is well-defined: We must show that $\phi_{t'}^i(\pi)$ is indeed in $\Pi^{n_{t'}=i}$. First observe that t is still constructed at position j since for π and $\phi_{t'}^i(\pi)$ the set of segments inserted until the j -th position (inclusive) is identical. Also notice that from this position on $\phi_{t'}^i(\pi)$ and π are identical, which implies that the set of trapezoids in the trapezoidal map that is constructed from now on, is identical for both permutations. Specifically, t remains the i -th last trapezoid that covers t' .

We still need to argue that t' is now constructed with the insertion of \bar{s} . Obviously, it cannot be constructed earlier since by definition \bar{s} is either the top or bottom segment. The important part is that the vertical walls that define t' are not blocked due to the new insertion order $\phi_{t'}^i(\pi)$; see also Figure 7. The only two segments that changed position are s and \bar{s} . The insertion of \bar{s} , which defines the top or bottom sides of t' , could block vertical walls. However, since its position in $\phi_{t'}^i(\pi)$ is later than its position in π it cannot block a wall that it did not block before. On the other hand, s , which is inserted earlier can only be a left segment that extends to the left or a right segment that extends to the right. Hence, it cannot intersect the vertical walls of t' at all. We conclude that t' is constructed with the insertion of \bar{s} and that $\phi_{t'}^i(\pi) \in \Pi^{n_{t'}=i}$.

$\phi_{t'}^i$ is injective: By definition of $\phi_{t'}^i(\pi)$ the i -th last trapezoid that covers t' is still t . Therefore, the inverse mapping of $\phi_{t'}^i(\pi)$ can be easily defined by interchanging the role of $left(t)$ with $top(t)$ and $right(t)$ with $bottom(t)$, respectively. \square

Corollary 12. *For a random element π of $\Pi^{n_{t'} \geq 1}$ the probability that $n_{t'}(\pi) = i$ for $i > 0$ is less than or equal to $1/2^{i-1}$.*

Proof. By Lemma 11 we know that $|\Pi^{n_{t'} > i}| \leq |\Pi^{n_{t'} = i}|$, adding $|\Pi^{n_{t'} > i}|$ to both sides we obtain

$$2|\Pi^{n_{t'} > i}| \leq |\Pi^{n_{t'} > i-1}|,$$

which implies

$$\begin{aligned} 2^{i-1}|\Pi^{n_{t'} > i-1}| &\leq |\Pi^{n_{t'} > 0}|, \\ 2^{i-1}|\Pi^{n_{t'} \geq i}| &\leq |\Pi^{n_{t'} \geq 1}|. \end{aligned}$$

And with $|\Pi^{n_{t'} = i}| \leq |\Pi^{n_{t'} \geq i}|$ we obtain

$$2^{i-1}|\Pi^{n_{t'} = i}| \leq 2^{i-1}|\Pi^{n_{t'} \geq i}| \leq |\Pi^{n_{t'} \geq 1}|.$$

Thus,

$$\Pr[n_{t'}(\pi) = i | \pi \in \Pi^{n_{t'} \geq 1}] = |\Pi^{n_{t'} = i}| / |\Pi^{n_{t'} \geq 1}| \leq 1/2^{i-1}$$

\square

Corollary 13. *For a random element π of $\Pi^{n_{t'} \geq i}$, the expected value for $n_{t'}(\pi)$ is constant.*

Proof.

$$\begin{aligned}
\mathbb{E}[n_{t'}(\pi) | \pi \in \Pi^{n_{t'} \geq 1}] &= \sum_{0 < i \leq n} i \cdot \Pr[n_{t'}(\pi) = i | \pi \in \Pi^{n_{t'} \geq 1}] \\
&\leq \sum_{0 < i \leq n} i \cdot \frac{1}{2^{i-1}} \\
&= 2 \cdot \sum_{0 < i \leq n} \frac{i}{2^i} \\
&\leq 4
\end{aligned}$$

□

According to Corollary 13 the expected number of different DAG trapezoids that cover t' is not more than 4. For every such t' that contains t' during the construction we may only get up to two bouncing nodes. By also taking into account two additional bouncing nodes that may occur at the destruction of t' , we can bound $\mathbb{E}[w_{t'} | \delta_{t'} = 1]$ by $2 \cdot 4 + 2 = 8$. Applying this to $\mathbb{E}[w_{t'} | \delta_{t'} = 1]$ in Equation 7 yields

$$O(n \log n) + 8 \sum_{t' \in \Delta} \mathbb{E}[\delta_{t'}], \quad (8)$$

which equals:

$$O(n \log n) + 8 \cdot \mathbb{E}\left[\sum_{t' \in \Delta} \delta_{t'}\right]. \quad (9)$$

Clearly, $\mathbb{E}[\sum_{t' \in \Delta} \delta_{t'}] = O(n \log n)$, since $\mathbb{E}[\sum_{t' \in \Delta} \delta_{t'}]$ is the expected number of trapezoids in \mathbb{T} , proving the following proposition:

Proposition 14. *Let S be a set of n pairwise interior disjoint x -monotone curves inducing a planar subdivision. Let \mathbb{G} be a DAG of linear size that was constructed by a randomized incremental insertion. The expected running time $f(n)$ of the recursive algorithm executed on \mathbb{G} is $O(n \log n)$.*

Our main theorem for this section follows immediately by plugging the value of $f(n)$ obtained in Proposition 14 into Theorem 10.

Theorem 15. *Let S be a set of n pairwise interior disjoint x -monotone curves inducing a planar subdivision. A point location data structure for S , which has $O(n)$ size and $O(\log n)$ query time in the worst case, can be built in expected $O(n \log n)$ time.*

5.2 An $O(n \log n)$ Verification Algorithm

Let $\mathcal{T}(S_i)$ denote the trapezoidal map obtained after inserting the first i curves. We also use this notation in order to identify the set of trapezoids of this map.

We denote by \mathcal{T}^* the collection of *all* trapezoids created during the construction of the DAG, including intermediate trapezoids that are killed by the insertion of later segments. More formally:

$$\mathcal{T}^* = \bigcup_{i=1}^n \mathcal{T}(S_i).$$

Let $\mathcal{A}(\mathcal{T}^*)$ denote the arrangement of all trapezoids in \mathcal{T}^* . Notice that a face of the arrangement may be covered by overlapping trapezoids. The *ply* of a point p in $\mathcal{A}(\mathcal{T}^*)$ is defined as the number of trapezoids in \mathcal{T}^* that cover p . The key to the improved algorithm is the following observation by Har-Peled [11].

Observation 16. *The length of a path in the DAG for a query point q is at most three times the ply of q in $\mathcal{A}(\mathcal{T}^*)$.*

It follows that we need to verify that the maximum ply of a point in $\mathcal{A}(\mathcal{T}^*)$ is $c_1 \log n$ for some constant $c_1 > 0$. We remark that this ply is established in an interior of a face of $\mathcal{A}(\mathcal{T}^*)$, since the longest path will always end in a leaf of the DAG, which, under the general position assumption, represents a trapezoid. Moreover, for any query point that falls on either a curve or an endpoint of the initial subdivision the search path will end in an internal node of the DAG. If, on the other hand, the query point q falls on a vertical edge of a trapezoid, the search path for q will be identical to a path for a query point in a neighboring trapezoid. Therefore, we consider the boundaries of the trapezoids as open.

Since the input curves are interior pairwise disjoint, according to the separation property deduced from [10], one can define a total order on the curves; see more details in Subsection 5.2.1 below. This order allows us to apply a modified version of an algorithm by Alt and Scharf [1], which originally detects the maximum ply in an arrangement of n axis-parallel rectangles in $O(n \log n)$ time. Recall that we only apply this verification algorithm on DAGs of linear size.

We would like to describe a linear space algorithm with $O(n \log n)$ running time for computing the ply of an arrangement of open trapezoids with the following properties: their bases are y -axis parallel (vertical walls) and if the top or bottom curves of two different trapezoids intersect not only in a joint endpoint then the two curves overlap completely in their joint x -range. The ply of such an arrangement is the maximum number of trapezoids containing a common point, that is, we are only interested in points located in faces of this arrangement. In Appendix C we restate the algorithm by Alt & Scharf [1] such that the general position assumption can be dropped. The algorithm constructs a balanced binary tree representing the possible x -intervals. It then performs a vertical sweep, recording the events of creation and destruction of a rectangle. The data is kept in the tree nodes, and a final traversal pushes the collected information to the leaves. The maximal ply will appear in one of the leaves.

Next, we define a reduction from the collection of open trapezoids \mathcal{T}^* to a collection \mathcal{R}^* of open axis-parallel rectangles such that the maximum ply in $\mathcal{A}(\mathcal{R}^*)$ is the same as the maximum ply in $\mathcal{A}(\mathcal{T}^*)$. Using this reduction we can finally describe a modification for the restated algorithm such that it can compute the ply of the arrangement of all trapezoids created during the construction of the DAG.

5.2.1 A Ply Preserving Reduction

Let \mathcal{T}^c be a collection of open trapezoids with y -axis parallel bases with the following property: if the top or bottom curves of two different trapezoids intersect not only in joint endpoints then the two curves overlap completely in their joint x -range. Let $\mathcal{A}(\mathcal{T}^c)$ denote the arrangement of the trapezoids in \mathcal{T}^c . Notice that each arrangement face can be covered by overlapping trapezoids. We describe a reduction from \mathcal{T}^c to \mathcal{R}^c , where \mathcal{R}^c is a collection of axis-parallel rectangles, such that the maximum ply in $\mathcal{A}(\mathcal{R}^c)$ equals to the maximum ply in $\mathcal{A}(\mathcal{T}^c)$.

In order to define the reduction we need to have a total order $<$ on the non-vertical curves of the trapezoids in \mathcal{T}^c , such that one can translate the curves one by one according to this order to $y = -\infty$ without hitting other curves that have not been moved yet. Guibas & Yao [10] defined an acyclic relation \prec on a set C of n interior disjoint x -monotone curves as follows:

Definition 17. For two such curves $cv_i, cv_j \in C$, let the open interval (a, b) be the x -range of cv_i and the open interval (c, d) be the x -range of cv_j . If $x\text{-range}(cv_i) \cap x\text{-range}(cv_j) \neq \emptyset$ then:
 $cv_i \prec cv_j \Leftrightarrow cv_i(x) < cv_j(x)$ for some $x \in x\text{-range}(cv_i) \cap x\text{-range}(cv_j)$.

As a matter of fact, their definition is more specific, in a way that the relation $cv_i \prec cv_j$ exists only if cv_i is the first curve encountered by cv_j in their joint x -range while translating cv_j to $y = -\infty$. In [10] it is also mentioned that \prec^+ , which is the transitive closure of \prec , is a partial order (as it allows transitivity). This partial order \prec^+ can be extended to a total order $<$ in many ways. One possible extension is defined as follows:

Definition 18. Let C be a set of interior disjoint x -monotone curves. For two curves $cv_i, cv_j \in C$, let the open interval (a, b) be the x -range of cv_i and the open interval (c, d) be the x -range of cv_j . The total order $<$ on C is defined as follows:
 $cv_i < cv_j \Leftrightarrow (cv_i \prec^+ cv_j) \text{ or } (\neg(cv_j \prec^+ cv_i) \text{ and } (cv_i \text{ left } cv_j))$
where $(cv_i \text{ left } cv_j)$ is true if the x -value of the left endpoint of cv_i is less than the x -value of the left endpoint of cv_j .

Clearly, if $cv_i \prec^+ cv_j$ is true then $cv_i < cv_j$ is true as well. If for two different curves cv_i, cv_j the expression $cv_j \prec^+ cv_i$ is true then obviously $cv_i \prec^+ cv_j$ is false and also the right-hand side expression in the “or” phrase is false, since $\neg(cv_j \prec^+ cv_i)$ is false. Therefore, $cv_i < cv_j$ is also false. If the partial order \prec^+ does not say anything about cv_i and cv_j then both $(cv_i \prec^+ cv_j)$

and $(cv_j \prec^+ cv_i)$ are false. Thus, $cv_i < cv_j$ will be true only if $(cv_i \text{ left } cv_j)$ is true.

Ottmann & Widmayer [18] presented a one-pass $O(n \log n)$ time algorithm for computing $<$, as in Definition 18, using linear space. Their algorithm performs a sweep using a horizontal line from bottom to top which stops at each endpoint of a curve. The data structure maintained by the algorithm represents the curves encountered so far in reverse order. When a bottom endpoint of a curve is met the curve is inserted into an auxiliary structure holding the active curves only. A curve is removed from the auxiliary structure when its top endpoint is met by the sweep line. Since we would like to translate the curves to $y = -\infty$, then we should only require the curves to be x -monotone. In addition, we can require the curves to be interior disjoint, rather than completely disjoint.

Definition 19. Let $\text{Rank}: C \rightarrow \{1, \dots, n\}$ denote a function returning the rank of a given x -monotone curve $cv \in C$ when sorting C according to the total order $<$ as defined above.

Definition 20. We define a reduction from \mathcal{T}^c to \mathcal{R}^c as follows; Every trapezoid $t \in \mathcal{T}^c$ is reduced to a rectangle $r \in \mathcal{R}^c$, such that:

- t and r have the same x -range,
i.e., $(\text{left}(t) = \text{left}(r))$ and $(\text{right}(t) = \text{right}(r))$, where left and right denote the left x -value and the right x -value of t (or r), respectively.
- $\text{top}(r)$ and $\text{bottom}(r)$ lie on $y = \text{Rank}(\text{top}(t))$ and $y = \text{Rank}(\text{bottom}(t))$, respectively.

Definition 20 provides a mapping from \mathcal{T}^c to \mathcal{R}^c , such that r is the rectangular region corresponding to t . In Appendix D we show that this mapping is bijective. We show there that the number of trapezoids in \mathcal{T}^c that cover a region a_t equals to the number of rectangles in \mathcal{R}^c that cover a_r , which is the region corresponding to a_t . In summary, we obtain the following theorem.

Theorem 21. Let \mathcal{T}^c be a collection of open trapezoids with the following properties: their bases are y -axis parallel (vertical walls) and if the top or bottom curves of two different trapezoids intersect not only in joint endpoints then the two curves overlap completely in their joint x -range. Let $\mathcal{A}(\mathcal{T}^c)$ denote the arrangement of the trapezoids in \mathcal{T}^c . Notice that each arrangement face can be covered by overlapping trapezoids. \mathcal{T}^c can be reduced to a collection of open axis-parallel rectangles \mathcal{R}^c , such that the maximum ply in $\mathcal{A}(\mathcal{R}^c)$ equals to the maximum ply in $\mathcal{A}(\mathcal{T}^c)$.

5.2.2 Modification of Alt & Scharf

Based on the correctness of the reduction described above we can extend the basic algorithm by Alt & Scharf [1] to support not only collections of axis-aligned rectangles but also collections of open trapezoids with y -axis parallel bases and non-intersecting top and bottom boundaries, if they intersect not only in joint

endpoints then they overlap completely in their joint x -range. The only part of the basic algorithm that should change is the top-to-bottom sweep. More precisely, the simple predicate that is used for sorting the y -events should be replaced with a new predicate that compares according to the reverse order of $<$, as given in Definition 18. The total order $<$ can be computed in a preprocessing phase using the algorithm in [18].

Notice that for simplicity we assumed that no two distinct endpoints in the original subdivision have the same x -value. However, if this is not the case, lexicographical comparison can be used on the endpoints of the curves in order to define the order of the induced vertical walls.

5.3 Summary

The two algorithms described in Subsection 5.1 and Subsection 5.2 can be used for defining efficient construction algorithms for static settings, according to the scheme presented in Theorem 10.

Using either verification algorithm, a construction algorithm with expected $O(n \log n)$ running time is obtained, implying the following main contribution of the paper:

Theorem 22. *Let S be a set of n pairwise interior disjoint x -monotone curves inducing a planar subdivision. A point location data structure for S , which has $O(n)$ size and $O(\log n)$ query time in the worst case, can be built in expected $O(n \log n)$ time.*

6 Conclusions and Open Problems

In this work we have described an optimal variant of a known algorithm for point location: the randomized incremental construction of the trapezoidal map. This fundamental point location algorithm supports general x -monotone curves and guarantees logarithmic query time and linear space. Previously with such guarantees, the expected construction time of the randomized search structure was $O(n \log^2 n)$, as was mentioned in [4]. Their construction algorithm uses an auxiliary algorithm for verifying that the maximal query path length is logarithmic, whose expected time complexity is $O(n \log^2 n)$. The latter dominates the overall construction complexity. However, we have proposed two novel verification algorithms—either of which could be used instead when constructing the search structure. These two efficient verification algorithms allow an expected $O(n \log n)$ construction time while having the same guarantees.

The two possible verification algorithms we have described can both be plugged into the suggested construction scheme. The first algorithm operates directly on the DAG \mathbb{G} and has a recursive nature. Its analysis relies on a bijection between all search paths in \mathbb{G} and those of the trapezoidal search tree \mathbb{T} . The second algorithm has a deterministic $O(n \log n)$ time complexity, and it is based on the computation of the maximal ply of all trapezoids that existed during the construction of the DAG \mathbb{G} . While the latter is deterministic, the

former does not require the construction of any other auxiliary structures and only uses the already constructed DAG.

Another contribution of this work, which in fact triggered the entire project, is the study of the fundamental difference between the length \mathcal{L} of the longest search path and the DAG depth \mathcal{D} , which is the length of the longest path in the constructed DAG. Clearly, efficiently computing the value of \mathcal{L} is not trivial, whereas \mathcal{D} can be easily accessed. We have clarified why the two entities are not trivially interchangeable and proved that the worst-case ratio of \mathcal{D}/\mathcal{L} is in $\Theta(n/\log n)$.

One major open problem, which is of theoretic interest, is whether it is sufficient to simply check the \mathcal{D} during the construction, as it is in fact done in the current CGAL implementation [12], and still expect a constant number of rebuilds. In other words, can we still expect a constant number of rebuilds if we just rely on \mathcal{D} , which is only an upper bound of \mathcal{L} ?

7 Acknowledgement

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A The Trapezoidal Search Tree \mathbb{T}

We have shown a bijection between all search paths in the DAG \mathbb{G} and those of the trapezoidal search tree \mathbb{T} (see Section 4). Using this bijection we were able to devise an efficient recursive verification algorithm that is described in Subsection 5.1. The analysis of the algorithm relies on the expected $O(n \log n)$ size of \mathbb{T} . Even though this bound seems to be folklore, we have not found the source in which it is actually proven. Therefore, we provide here a proof.

Definition 23. *Let S be a set of n pairwise interior disjoint x -monotone curves inducing a planar subdivision. The trapezoidal search tree \mathbb{T} for S is a full binary tree constructed as the DAG \mathbb{G} using the same insertion order while skipping the merge step.*

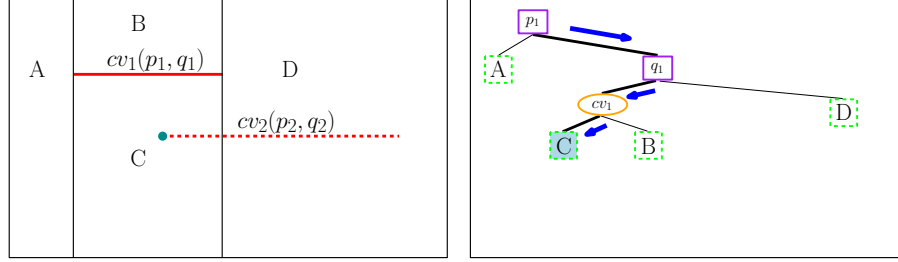
The following lemma bounds the expected size of \mathbb{T} to be $O(n \log n)$.

Lemma 2 *Let S be a set of n pairwise interior disjoint x -monotone curves inducing a planar subdivision. The expected number of leaves in the trapezoidal search tree \mathbb{T} , which is constructed as the DAG but without merges, is $O(n \log n)$.*

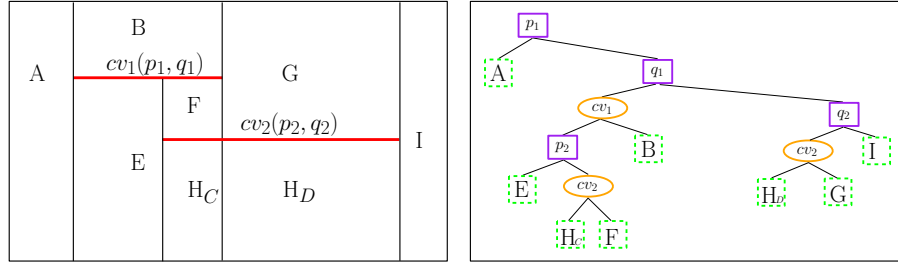
Proof. We would like to bound the expected number of leaves in \mathbb{T} , namely, the expected number of trapezoids in the decomposition without merges. To ease the argument, we can symbolically shorten every curve at its two endpoints by an arbitrarily small value $\varepsilon > 0$. In other words, if a curve cv has an x -range (a, b) , then the shortened curve will have an x -range $(a + \varepsilon, b - \varepsilon)$. The curves of the updated subdivision are now completely disjoint. This operation only gives rise to new artificial trapezoids. Hence, it is sufficient to bound the expected number of trapezoids in this subdivision of shortened curves.

Now we would like to bound the number of trapezoids in the set of shortened curves. It is clearly bounded by the number of vertical edges plus 1. First, consider the vertical line W through one endpoint of a curve cv . W is intersected by m curves. Suppose that cv is the i th inserted curve among these m curves. The $i - 1$ already inserted curves partition W into i intervals. However, we are only interested in the interval I containing the endpoint of cv , as it will appear in the final structure. Curves inserted after cv may split I . The expected number of intersections in I (including the endpoint of cv) is $O((m - i)/i)$. The probability that cv will be inserted i th is $\frac{1}{m}$. Summing up over all possible insertion orders we get that the expected number of intersections in I is $\sum_{i=1}^m \frac{1}{m} \cdot \frac{(m-i)}{i} = O(\log m)$.

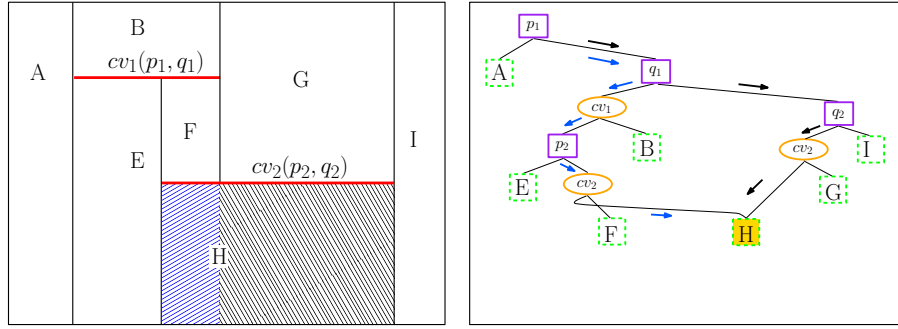
However, since $m \leq n$, this number can be bounded $O(\log n)$. There are $O(n)$ vertical walls, giving a total of expected $O(n \log n)$ intersections. Thus, the expected number of vertical edges is $O(n \log n)$ as well, and, clearly, this is also the expected number of leaves in the tree. \square



(a)



(b)



(c)

Figure 1: Updating the DAG with a second curve $cv_2(p_2, q_2)$. (a) Locating p_2 (the left endpoint of cv_2) in the DAG of the trapezoidal map for $cv_1(p_1, q_1)$. The query path is highlighted with blue arrows. (b) Due to the insertion of cv_2 , trapezoids C and D are split into trapezoids E, F, H_C and H_D, G, I , respectively. The obtained structure is in fact the Trapezoidal Search Tree \mathbb{T} for cv_1, cv_2 . (c) Newly created trapezoids H_C, H_D are merged into trapezoid H , since cv_2 blocks the wall induced by q_1 . The resulting DAG contains two directed paths (marked) from the root to the leaf labeled H , which are both valid search paths.

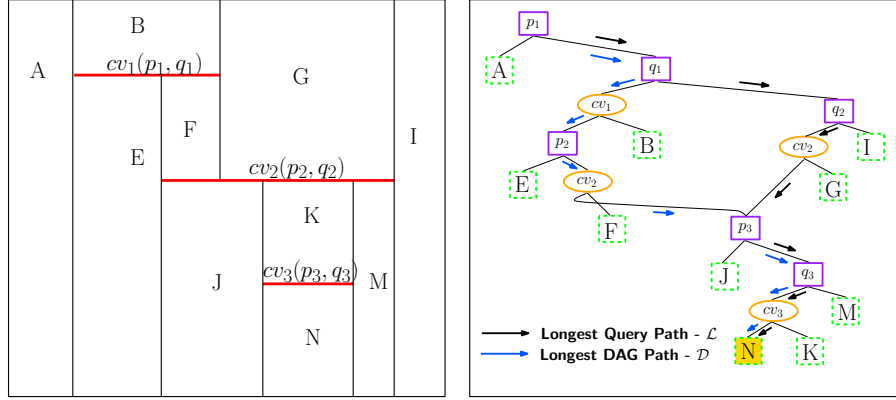


Figure 2: The trapezoidal map and corresponding DAG after inserting $cv_3(p_3, q_3)$ to the subdivision from Figure 1. The leaf representing trapezoid H in the former structure is replaced with a subtree rooted at p_3 . There are two directed paths starting at the root that reach trapezoid N , marked in black and blue. The black represents the longest query path, and the blue represents the longest DAG path. The black path is the search path for all queries that end up in trapezoid N . The blue path is not a valid search path, since all points in N are to the right of q_1 , that is, such a query would never visit the left child of q_1 . This scenario occurs due to the merge that was part of the insertion of cv_2 (see Figure 1(c)) creating two different paths to a leaf, which became the inner node p_3 in the updated structure.

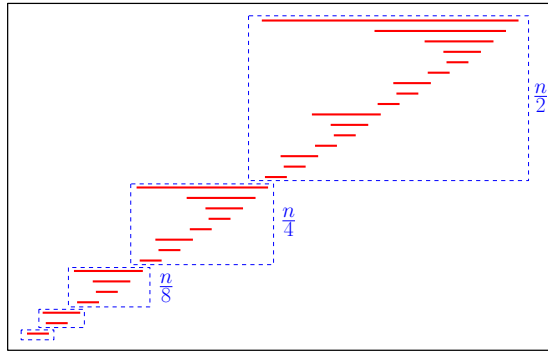


Figure 3: A recursive construction establishing the $\Omega(n/\log n)$ lower bound for the \mathcal{D}/\mathcal{L} ratio.

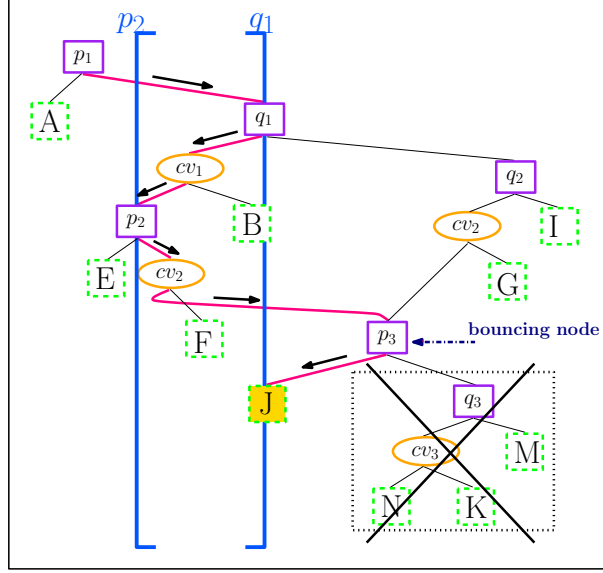


Figure 4: The node p_3 is a **bouncing node** for the path leading into the left part of trapezoid J (see also Figure 2). The history interval when reaching node p_3 is (p_2, q_1) . The decision at p_3 is already predetermined by the history of the path: q_1 is to the left of p_3 , the fact that the path descended to the left at node q_1 implies that it also descends to the left at p_3 .

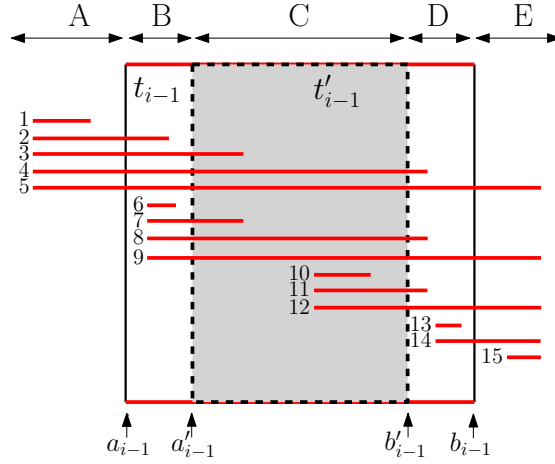


Figure 5: Possible positions for cv_i in relation to trapezoid t_{i-1} in \mathbb{G}_{i-1} , which covers trapezoid t'_{i-1} in \mathbb{T}_{i-1} .

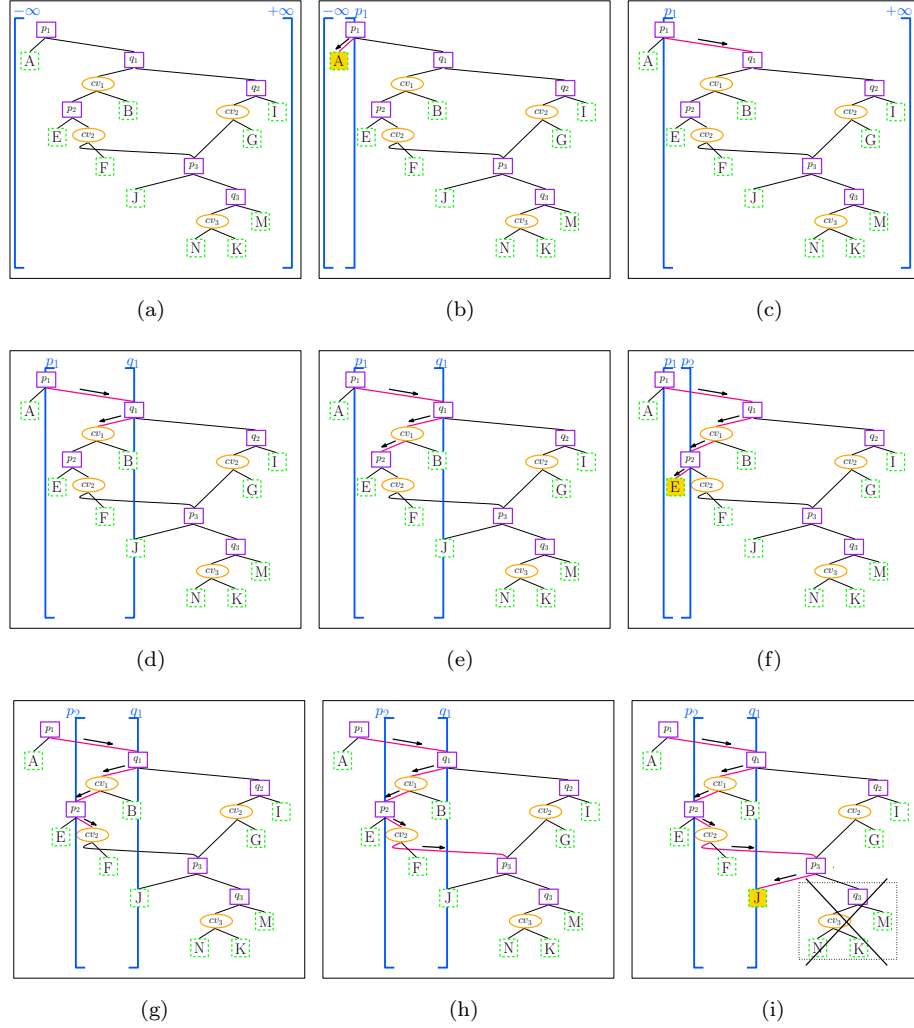


Figure 6: The first 9 steps of the recursive verification algorithm run on the search structure for 3 curves, as illustrated in Figure 2. The interval of possible x -values is marked by the blue brackets. In each step the growing path so far is marked with arrows. In (i) the interval of possible x -values remains $[p_2, q_1]$ and does not shrink since p_3 is not contained in it. The subgraph rooted at the right child of p_3 is clearly not contained in $[p_2, q_1]$, since it represents regions that are completely to the right of p_3 , and is, therefore, skipped. p_3 is a bouncing node for the path depicted in (i).

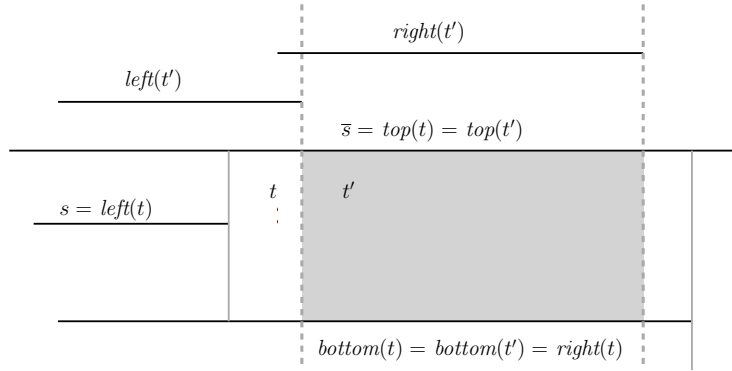


Figure 7: Example configuration of t' covered by t . In this case $right(t)$ is also $bottom(t) = bottom(t')$ as it extends to the left. Hence, s must be $left(t)$. One possible insertion order π of the segments causing this configuration is: $right(t')$, $left(t')$, $top(t)$, $bottom(t)$, $s = left(t)$. Note that t' is created with the insertion of $bottom(t)$, whereas t is created afterwards, i. e., with the insertion of $s = left(t)$. Now, $\phi_{t'}^i$ swaps s with $\bar{s} = top(t) = top(t')$. At its new insertion position in $\phi_{t'}^i(\pi)$ the segment \bar{s} cannot block the vertical walls (dashed) induced by $left(t')$ and $right(t')$ as it did not do so at its earlier position in π . On the other hand, the segment s , which is now inserted earlier, extends to the left and cannot block these walls at all. Hence, according to $\phi_{t'}^i(\pi)$ the trapezoids t and t' are created simultaneously, namely with the insertion of \bar{s} .

B Bijection between Search Paths in \mathbb{G} and \mathbb{T}

We provide here the full case analysis that is needed for proving Proposition 4.

Proposition 1 *Let S be a set of n pairwise interior disjoint x -monotone curves inducing a planar subdivision. Let \mathbb{G} and \mathbb{T} be the DAG and the trapezoidal search tree created using the same permutation of the curves in S , respectively. There exists a canonical bijection among all search paths in \mathbb{G} and those of \mathbb{T} , that is, for any query point q , the corresponding search paths for q in \mathbb{G} and \mathbb{T} are identical up to bouncing nodes.*

Proof. Let q be a query point and t and t' be the leaf trapezoids containing q in \mathbb{G} and \mathbb{T} , respectively. Obviously, the top and bottom curves of t and t' are identical and t covers t' since merges are only allowed in \mathbb{G} . Suppose that while searching for q we maintain the history-interval of possible x -values². At the end of the search in \mathbb{T} this interval is obviously identical to the x -range of t' . We show by induction the bijection between the two search paths for q and in fact we also show that the history intervals maintained while searching in \mathbb{G} and \mathbb{T} are identical, i.e., the history interval that is eventually obtained by the search in \mathbb{G} is identical to the x -interval of t' .

Let $\mathbb{G}_i, \mathbb{T}_i$ denote the DAG and the trapezoidal search tree after the first i curves were inserted, respectively. We denote by t_i and t'_i the trapezoids containing the query point q in \mathbb{G}_i and \mathbb{T}_i , respectively. Let (a_i, b_i) and (a'_i, b'_i) denote the x -intervals of t_i in \mathbb{G}_i and t'_i in \mathbb{T}_i , respectively. The base case is trivial since $\mathbb{G}_1 = \mathbb{T}_1$. Now suppose that the statement holds for $i - 1$. We show that it holds for i as well. The i th curve $cv_i(p_i, q_i)$ is now inserted into both \mathbb{G}_{i-1} and \mathbb{T}_{i-1} . The basic argument is as follows. For both endpoints of cv_i there are essentially three cases: (i) the point is outside t_{i-1} and t'_{i-1} : the point has no effect on both paths. (ii) the point is inside t_{i-1} and t'_{i-1} : the point shows up as a normal node in both paths and the history intervals are updated accordingly. (iii) the point is inside t_{i-1} but not in t'_{i-1} : the point has no effect on the search path in \mathbb{T}_i while it may show up on the search path in \mathbb{G}_i , but only as a bouncing node, i.e., the history interval remains unchanged. Figure 8 shows the 15 possible positions to insert cv_i with respect to t_{i-1} and t'_{i-1} . We denote the five different vertical slabs (regions) depicted in the figure by A, B, C, D , and E . Note that regions B and D may have zero width.

For ease of reading we group the optional positions according to the region containing p_i as follows:

- p_i is located at region A (positions 1-5 in Figure 8). For these positions p_i will not be added to the path to q in either \mathbb{G}_i or \mathbb{T}_i (case (i)). We now distinguish the different cases depending on the position of q_i .

²Since we do not require the points to be in general position we consider the lexicographic order of the points. Therefore, this interval which is referred to as x -interval is essentially defined by the x, y coordinates of two points

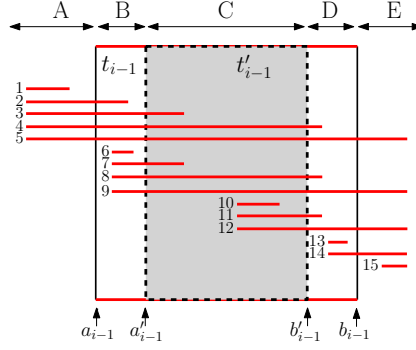


Figure 8: Possible positions for cv_i in relation to trapezoid t_{i-1} in \mathbb{G}_{i-1} , which covers trapezoid t'_{i-1} in \mathbb{T}_{i-1} .

- Position 1: q_i lies in region A as well and, therefore, will not affect both paths (case (i)). Clearly, $t_i = t_{i-1}$ and $t'_i = t'_{i-1}$.
 - Position 2: q_i lies in region B (case (iii)). q_i will not be added to the path to q in \mathbb{T}_i . However, it will be added to the path in \mathbb{G}_i but only as a bouncing node for this path. The x -interval maintained during the search in \mathbb{G}_i will not be affected.
 - Position 3: q_i lies in region C (case (ii)). The search paths for q in both \mathbb{G}_i and \mathbb{T}_i will include q_i . If q is in the x -range of cv_i then an additional internal node representing cv_i will appear in the path for q in both structures. The interval (a'_i, b'_i) in such a case would be (a'_{i-1}, q_i) . If, on the other hand, q is to the left of q_i then the new interval would be (q_i, b'_{i-1}) .
 - Position 4: q_i lies in region D (case (iii)). Similar to the case where q_i lies in region B . In addition, since cv_i intersects t'_{i-1} completely, an internal node cv_i will be added to both structures.
 - Position 5: q_i lies in region E and, therefore, will not affect both paths (case (i)). Since cv_i intersects t'_{i-1} completely, an internal node cv_i will be added to both structures.
- p_i is located in region B (positions 6-9 in Figure 8). For these positions p_i will not be added to the path to q in \mathbb{T}_i . However, it will be added to the path in \mathbb{G}_i but only as a bouncing node for this path, since it is not contained in (a'_{i-1}, b'_{i-1}) (case (iii)). We now distinguish the several cases depending on the position of q_i .
 - Position 6: q_i lies in region B (case (iii)). In such a case q_i will be added to the query path to q as a bouncing node in \mathbb{G}_i , but will not affect the interval maintained during the search since q_i is not contained in (a'_{i-1}, b'_{i-1}) . The query path to q in \mathbb{T}_i will not change, since cv_i does not intersect t'_{i-1} .

- Position 7: q_i lies in region C (case (ii)). The search paths for q in both \mathbb{G}_i and \mathbb{T}_i will include q_i . If q is in the x -range of cv_i then an additional internal node representing cv_i will appear in the path for q in both structures. The interval (a'_i, b'_i) in such a case would be (a'_{i-1}, q_i) . If, on the other hand, q is to the left of q_i then the new interval would be (q_i, b'_{i-1}) .
- Position 8: q_i lies in region D (case (iii)). Similar to the case where q_i lies in region B . In addition, since cv_i intersects t'_{i-1} completely, an internal node cv_i will be added to both structures.
- Position 9: q_i lies in region E and, therefore, will not affect both paths (case (i)). Since cv_i intersects t'_{i-1} completely, an internal node cv_i will be added to both structures.
- p_i is located inside in region C (positions 10-12 in Figure 8). In these positions p_i will be added to the search path of a query point q that lies in t'_{i-1} both in \mathbb{G}_i and in \mathbb{T}_i , since it is contained in (a'_{i-1}, b'_{i-1}) (case (ii)). We now distinguish the several cases depending on the position of q_i .
 - Position 10: q_i lies in region C (case (ii)). cv_i is contained completely in region C . The same internal nodes, depending on the position of q , will be added for both search structures.
 - Position 11: q_i lies in region D (case (iii)). If the query point q is located to the left of p_i then no new node (other than p_i) will be added to the search paths of q in both \mathbb{G}_i and \mathbb{T}_i . If, on the other hand, q is in the x -range of cv_i then q_i will be added to the path as a bouncing node in \mathbb{G}_i , but will not appear in \mathbb{T}_i . In addition the paths in the two structures will be added with a node representing cv_i .
 - Position 12: q_i lies in region E and, therefore, will not affect both paths (case (i)). Depending on the location of q , an internal node cv_i may be added to the paths in both structures.
- p_i is located in region D (positions 13-14 in Figure 8). For these positions p_i will not be added to path to q in \mathbb{T}_i . However, it will be added to the path in \mathbb{G}_i but only as a bouncing node for this path (case (iii)). In both positions q_i is to the right of p_i and is, therefore, blocked by p_i for query points that lie in t'_{i-1} and will not be added to the search paths of such points.
- p_i is located to the right of t_{i-1} , that is, in region E (case (i)). In Figure 8 the relevant position is 15. Both p_i and q_i , which is located to the right of p_i , will not affect the search paths for q in both structures.

In each of these 15 different cases, whenever p_i or q_i are only added to \mathbb{G}_i , the node will appear as a bouncing node for the path to q .

□

C An Algorithm for Computing the Ply of an Arrangement of Axis-aligned Rectangles

The algorithm of Alt & Scharf [1] is an $O(n \log n)$ algorithm that computes the maximum ply of an arrangement of axis-aligned rectangles in general position, using $O(n)$ space. We present a minor modification, which does not assume general position, i.e., rectangles may share boundaries. Moreover, it can consider each of the four boundaries of a rectangle as either belonging to the rectangle or not; we call these closed or open boundaries, respectively.

Given a set of n axis-aligned rectangles, let x_1, x_2, \dots, x_k , $k \leq 2n$ be the sorted set of x -coordinates of the vertical sides of the rectangles. The ordered set of intervals \mathcal{I} is defined as follows; for $i \in 1, 2, \dots, k-1$, the $2(i-1)$ th and $2(i-1)+1$ st intervals in the set \mathcal{I} are $[x_i, x_i]$ and (x_i, x_{i+1}) , respectively. The last interval is $[x_k, x_k]$. A balanced binary tree T is then constructed, holding all intervals in \mathcal{I} in its leaves, according to their order in \mathcal{I} . An internal node represents the union of the intervals of its two children, which is a contiguous interval. In addition, each internal node v stores in a variable $v.x$ the x -value of the merge point between the intervals of its two children. Since we extended the algorithm to support both open or closed boundaries, internal nodes also maintain a flag indicating whether the merge point is to the left or to the right of the x -value.

According to the description of the algorithm in [1], a sweep is performed using a horizontal line from $y = \infty$ to $y = -\infty$. The sweep-line events occur when a rectangle starts or ends, i.e., when top or bottom boundary of a rectangle is reached. Since the rectangles are not in general position, several events may share the same y -coordinate. In such a case, the order of event processing in each y -coordinate is as follows:

1. Closing rectangle with open bottom boundary events.
2. Opening rectangle with closed top boundary events.
3. Closing rectangle with closed bottom boundary events.
4. Opening rectangle with open top boundary events.

The order of event processing within each of these four groups in a specific y -coordinate is not important.

The basic idea of the algorithm is that each sweep event updates the leaves of the tree T that span the intervals that are covered by the event. Therefore, each leaf holds a counter c for the number of covering rectangles in the current position of the horizontal sweep line. In addition, each leaf maintains in a variable c_m the maximal number of covering rectangles for this leaf seen so far. Clearly, the maximal coverage of an interval is the maximal c_m of all leaves. The problem with this naïve approach is that one such update can already take $O(n)$ time. Therefore, the key idea of [1] is that when updating an event of a rectangle whose x -range is (a, b) , one should follow only two paths; the path

to a and the path to b . The nodes on the path should hold the information of how to update the interval spanned by their children. In the end of the update the union of intervals spanned by the updated nodes (internal nodes and only 2 leaves) is (a, b) .

In order to hold the information in the internal nodes each internal node should maintain the following variables:

- l A counter storing the difference between the number of rectangles that were opened and that were closed since the last traversal of the left child of v and that cover the interval spanned by that child.
- r A counter storing the difference between the number of rectangles that were opened and that were closed since the last traversal of the right child of v and that cover the interval spanned by that child.
- l_m A counter storing the maximum value of l since the last traversal of that child.
- r_m A counter storing the maximum value of r since the last traversal of that child.

A leaf, on the other hand, holds two variables:

- c The coverage of the associated interval during the sweep at the point the leaf was traversed for the last time.
- c_m The maximum coverage of the associated interval during the sweep from the start until the leaf was traversed for the last time.

In relation to these values we define the following functions:

$$t(v) = \begin{cases} u.l + t(u) & \text{if } v \text{ is the left child of } u \\ u.r + t(u) & \text{if } v \text{ is the right child of } u \\ 0 & \text{if } v \text{ is the root} \end{cases},$$

$$t_m(v) = \begin{cases} \max(u.l_m, u.l + t_m(u)) & \text{if } v \text{ is the left child of } u \\ \max(u.r_m, u.r + t_m(u)) & \text{if } v \text{ is the right child of } u \\ 0 & \text{if } v \text{ is the root} \end{cases}.$$

At any point of the sweep the following two invariants hold for every leaf ℓ and its associated interval I :

- The current coverage of I is: $\ell.c + t(\ell)$.
- The maximum coverage of I that was seen so far is: $\max(\ell.c_m, \ell.c + t_m(\ell))$.

Updating the structure with an event is done as follows: Let I be the x -interval spanned by the processed rectangle creating the event. Depending on whether the rectangle starts or ends, we set a variable $d = 1$ or $d = -1$, respectively. We follow the two search paths to the leftmost leaf and the rightmost leaf that

are covered by I . In the beginning the two paths are joined until they split, for every node w on this path (including the split node) we can ignore d and simply update the tuple $(w.l, w.r, w.l_m, w.r_m)$ using $t(w)$ and $t_m(w)$ according to the invariants stated above. Note that this process needs to clear the corresponding values in the parent node as otherwise the invariants would be violated.³ After the split the paths are processed separately. We discuss here the left path, the behavior for the right path is symmetric. Let v be a node on the left path. As long as v is not a leaf we update $(v.l, v.r, v.l_m, v.r_m)$ as usual. However, if the path continues to the left we also have to incorporate d into $v.r$ and $v.r_m$ as the subtree to the right is covered by I . If v is a leaf we simply update $v.c$ and $v.c_m$ using $t(v)$, $t_m(v)$ and d . A more detailed description (including pseudo code) can be found in [1]. In total, this process takes $O(\log n)$ time.

Finally, in order to find the maximal number of rectangles covering an interval one last propagation from root to leaves is needed, such that all l, r, l_m, r_m values of internal nodes are cleared. This is done using one traversal on T . Now, the maximal number of rectangles covering an interval is the maximal c_m of all leaves of T .

Clearly, the running time of the algorithm is $O(n \log n)$, since constructing the tree and sorting the y -events takes $O(n \log n)$ time. Updating each of the $2n$ y -events takes $O(\log n)$ time, and the final propagation of values to the leaves takes $O(n)$ time. The algorithm uses $O(n)$ space.

³Notice that using $t(w)$ and $t_m(w)$ here takes constant time since we only need to access the parent node as all previous nodes on the path towards the root are already processed.

D Bijection between the Trapezoids in \mathcal{T}^c and the Rectangles in \mathcal{R}^c

We devise here a proof for Theorem 21, claiming that the mapping from \mathcal{T}^c to \mathcal{R}^c , presented in Definition 20, is bijective.

Given a subdivision, one can partition the plane into vertical slabs by passing a vertical line through every endpoint of the subdivision, and then partition each slab into regions by intersecting it with all the curves in the subdivision. This defines a decomposition of the plane into at most $2(n+1)^2$ regions (see [4], for example).

Lemma 24. *Let $\text{Regions}(\text{arr})$ denote the collection of regions of an arrangement arr , as defined above. For any region $a_t \in \text{Regions}(\mathcal{A}(\mathcal{T}^c))$ let $a_r \in \text{Regions}(\mathcal{A}(\mathcal{R}^c))$ be the rectangular region corresponding to a_t . The collection $\text{Regions}(\mathcal{A}(\mathcal{R}^c))$ of all such rectangular regions spans the plane.*

Proof. Trivial. The slabs remain the same and within each slab the rectangular regions remain adjacent. \square

Lemma 25. *Let $a_t \in \text{Regions}(\mathcal{A}(\mathcal{T}^c))$ be a region and let $a_r \in \text{Regions}(\mathcal{A}(\mathcal{R}^c))$ be the rectangular region corresponding to a_t . The number of rectangles in \mathcal{R}^c that cover a_r is at least the number of trapezoids in \mathcal{T}^c that cover a_t . In other words, for every $t \in \mathcal{T}^c$ that covers a_t its corresponding rectangle $r \in \mathcal{R}^c$ covers a_r .*

Proof. Let $\{t_1, t_2, \dots, t_m\} \subseteq \mathcal{T}^c$ be the set of trapezoids, ordered by creation time, such that for every $i \in \{1, \dots, m\}$, t_i covers a_t . Let $\{r_1, r_2, \dots, r_m\} \subseteq \mathcal{R}^c$ be the set of corresponding rectangles, such that r_i corresponds to t_i for $i \in \{1, \dots, m\}$. For any t_i , since t_i covers a_t we get that $x\text{-range}(a_t) \subseteq x\text{-range}(t_i)$. By Definition 20 the x -ranges remain the same after the reduction, and therefore $x\text{-range}(a_r) \subseteq x\text{-range}(r_i)$. Since t_i covers a_t then we also get that in the shared x -range $\text{top}(t_i)$ is above or on $\text{top}(a_t)$ and $\text{bottom}(t_i)$ is below or on $\text{bottom}(a_t)$. According to Definition 20, it immediately follows that $\text{Rank}(\text{top}(t_i)) \geq \text{Rank}(\text{top}(a_t))$. In other words, $\text{top}(r_i)$ is above or on $\text{top}(a_r)$. Similarly, $\text{bottom}(r_i)$ is below or on $\text{bottom}(a_r)$. We conclude that r_i covers a_r . \square

Lemma 26. *Let $a_r \in \text{Regions}(\mathcal{A}(\mathcal{R}^c))$ be a rectangular region, whose corresponding region is $a_t \in \text{Regions}(\mathcal{A}(\mathcal{T}^c))$. The number of trapezoids in \mathcal{T}^c that cover a_t is at least the number of rectangles in \mathcal{R}^c that cover a_r . In other words, for every $r \in \mathcal{R}^c$ that covers a_r its corresponding trapezoid $t \in \mathcal{T}^c$ covers a_t .*

Proof. Let $\{r_1, r_2, \dots, r_m\} \subseteq \mathcal{R}^c$ be the set of rectangles, such that for every $i \in \{1, \dots, m\}$, r_i covers a_r . Let $\{t_1, t_2, \dots, t_m\} \subseteq \mathcal{T}^c$ be the set of corresponding trapezoids, such that t_i corresponds to r_i for $i \in \{1, \dots, m\}$. Proving that for any $i \in \{1, \dots, m\}$, t_i covers a_t , is done symmetrically to the proof of Lemma 25. \square

Combining Lemma 25 and Lemma 26 we conclude that the number of trapezoids in \mathcal{T}^c that cover a region a_t equals to the number of rectangles in \mathcal{R}^c that cover a_r , which is the corresponding region to a_t . The covering rectangles are the reduced trapezoids in the set of trapezoids covering a_t . Since both $\text{Regions}(\mathcal{A}(\mathcal{T}^c))$ and $\text{Regions}(\mathcal{A}(\mathcal{R}^c))$ span the plane (Lemma 24), we obtain Theorem 21.